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A GENERAL COST STRUCTURE FOR THE SEMI-MARKOV SHOCK MODEL.(U)
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A GENERAL COST STRUCTURE FOR
THE SEMI-MARKOV SHOCK MODEL

by

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ABSTRACT

The semi-Markov shock model represents a system where shocks occur at random points in time causing random magnitudes of damage in such a way that the cumulative damage to the system over time is a semi-Markov process. System failure can occur at any of the shock times and the probability of a failure is a function of the cumulative damage. An optimal control limit type policy has previously been derived assuming that costs are incurred at the replacement time. This note is to show that the inclusion of state dependent maintenance costs does not significantly affect the form of the optimal control limit.

KEY WORDS: Optimal Replacement, Semi-Markov Processes, Markov Renewal Theory

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1. INTRODUCTION AND NOTATION

The optimal control limit for a semi-Markov shock model has been derived in Feldman (1976). In that paper, a very simple cost function was used. The purpose of this note is to extend the results of that paper to include a general cost structure.

Although a complete description of the process is given in the previous paper, it will be repeated here for ease of presentation. Let the Markov renewal process $(\hat{X},\hat{T}) = \{\hat{X}_n,\hat{T}_n;\ n=0,1,\ldots\}$ be a non-terminating process with state space E where $\hat{T}_0, \hat{T}_1, \hat{T}_2, \ldots$ denotes the initial, first, second, . . . shock times and $\hat{X}_0, \hat{X}_1, \hat{X}_2, \ldots$ denotes the cumulative damage at the respective shock times. (The initial shock time is always zero, i.e., $T_0=0$.) The state space E is taken to be a subset of the non-negative reals or the non-negative integers. Let $h(x)$ denote the probability that the system fails at time $\hat{T}_n$ given $\hat{X}_n = x$ and given that the system was working at time $\hat{T}_{n-1}$. The function h is assumed to be non-decreasing, and it is assumed that there exists $y \geq 0$ and some $n = 0,1,\ldots$ such that $h(y) > 0$ and $P\{\hat{X}_n \leq y \mid \hat{X}_0 = x\} < 1$ for all $x \leq y$. (This last assumption is used to insure a finite expected failure time.)

Define the random variable L to be the n such that $\hat{T}_n$ is the failure time of the system; thus,

$$P\{L > n \mid \hat{X}_0,\hat{X}_1,\ldots,\hat{X}_n\} = (1-h(\hat{X}_1))\ldots(1-h(\hat{X}_n)). \quad (1)$$

The time to failure is defined by

$$\zeta(\omega) = \hat{T}_{L(\omega)}(\omega). \quad (2)$$

Let the "death state" Δ be a distinct point not in E and define the Markov renewal process (X,T) by

$$X_n(\omega) = \begin{cases} \hat{X}_n(\omega) & \text{if } n < L(\omega), \\ \Delta & \text{if } n \geq L(\omega); \end{cases} \tag{3}$$

and

$$T_n(\omega) = \begin{cases} \hat{T}_n(\omega) & \text{if } n \leq L(\omega), \\ +\infty & \text{if } n > L(\omega). \end{cases} \tag{4}$$

The semi-Markov kernel of (X,T) is denoted by Q and is defined for $x, y \in E$, $t \geq 0$, by

$$\begin{aligned} Q(x,y,t) &= P\{X_1 \leq y,\ T_1 \leq t \mid X_0 = x\} \\ &\equiv P_x\{X_1 \leq y_1,\ T_1 \leq t\} \\ &= P_x\{\hat{X}_1 \leq y,\ \hat{T}_1 \leq t,\ L>1\} \\ &= \int_{u \in [0,y]} \hat{Q}(x,du,t)\,(1-h(u)), \end{aligned} \tag{5}$$

where $\hat{Q}$ is the semi-Markov kernel of $(\hat{X},\hat{T})$ and should be a given quantity. The Markov renewal kernel associated with Q is R and is defined, for $x, y \in E$, $t \geq 0$, by

$$R(x,y,t) = \sum_{n=0}^{\infty} Q^n(x,y,t) \tag{6}$$

where

$$Q^{n+1}(x,y,t) = \int_{u\in E} \int_{s\in[0,t]} Q(x,du,ds)Q^n(x,y,t-s).$$

The imbedded Markov chain X will have transition matrix P and, by an abuse of notation, its potential will be R; thus

$$P(x,y) = \lim_{t\to\infty} Q(x,y,t) \tag{7}$$

and

$$R(x,y) = \lim_{t\to\infty} R(x,y,t) = \sum_{n=0}^{\infty} P^n(x,y). \tag{8}$$

For a fixed control limit α, let τ_α denote the replacement time and let Λ_α denote the index of the shock time at which replacement occurs; that is,

$$\Lambda_\alpha(\omega) = L(\omega) \wedge \inf\{n: X_n \geq \alpha\} \tag{9}$$

and

$$\tau_\alpha(\omega) = T_{\Lambda_\alpha}(\omega). \tag{10}$$

Equations (3) and (4) define the shock model and equations (9) and (10) define the replacement policies. In the next section the cost structure is given and in Section 3 it is shown that previous results apply to this more general situation.

2. THE COST STRUCTURE

The costs used in Feldman (1976) were a fixed replacement cost and an additional incremental cost incurred upon failure. In Feldman (1977), a general replacement cost function was used but again no maintenance cost was considered. Optimality was based on long run average cost assuming instantaneous replacement with an identical system. In this note, both a maintenance and a replacement cost function are used. Furthermore, it is assumed that the time for replacement is a random variable denoted by σ.

Consider the renewal process formed by repeated replacements of identical systems, each system having a lifetime given by the distribution of τ_α and a replacement time given by the distribution of σ. Let the cost of the first, second, . . . replaced system be $C_1, C_2, \ldots$, where $\{C_1, C_2, \ldots\}$ is a sequence of independent identically distributed random variables. If N_t is the number of renewals in $[0,t]$ and ψ_α is the long-term cost per unit time, then

$$\psi_\alpha = \lim_{t\to\infty} \frac{1}{t}[C_1 + \ldots + C_{N_t}] = \frac{E[C_1]}{E[\tau_\alpha+\sigma]} . \tag{11}$$

Let the cost of replacement be given by the function f, where f(x) is the replacement cost if the cumulative damage is x at replacement time. Let the maintenance cost be determined by the rate function c(x), where c(x) is the cost per unit time incurred while the system remains in state x. Thus the expected cost per system is defined by

$$E[C_1] = E[f(X_{\Lambda_\alpha}) + \sum_{n=0}^{\Lambda_\alpha - 1} c(X_n)(T_{n+1} - T_n)] . \tag{12}$$

The objective is to find α^* such that α^* minimizes ψ_α where ψ_α is defined by equations (11) and (12).

3. OPTIMAL REPLACEMENT

The main trick is to substitute the maintenance and replacement cost functions by a single equivalent replacement cost function. This is not a new trick and was done by Dynkin (1963) for Markov processes and has been used often for both Markov chains and Markov processes. The following lemma incorporates these ideas for Markov renewal processes.

(3.1) LEMMA: Let (X,T) be the Markov renewal process defined by equations (3) and (4) and let Λ_α be defined by equation (10). Then

$$E_x\left[\sum_{n=0}^{\Lambda_\alpha - 1} c(X_n)(T_{n+1}-T_n)\right] = \int_{y\in E} R(x,dy)c(y)m(y) - E_x\left[\int_{y\in E} R(X_{\Lambda_\alpha},y)c(y)m(y)\right],$$

where $m(x)$ is the mean sojourn time in state x; or

$$m(x) = \int_{t\geq 0} [1-\hat{Q}(x,E,t)]\, dt \ .$$

Proof: Let ϕ be an arbitrary non-negative function and let $R\phi(x) = \int_{y\in E} R(x,dy)\phi(y)$. Then for the Markov chain X,

$$R\phi(x) = E_x\left[\sum_{n=0}^{\Lambda-1} \phi(x_n)\right] + E_x[R\phi(X_\Lambda)] \tag{13}$$

(see Çinlar (1975, page 201)). For ease of notation, let $W_n = T_{n+1} - T_n$

and define $\phi(x) = c(x)m(x)$. Now note that

$$R\phi(x) = E_x[\sum_{n=0}^{\infty} c(X_n)m(x)]$$

$$= \sum_{n=0}^{\infty} E_x[c(X_n)E_x[W_n|X_n]]$$

$$= \sum_{n=0}^{\infty} E_x[E_x[c(X_n)W_n|X_n]]$$

$$= \sum_{n=0}^{\infty} E_x[c(X_n)\cdot W_n]$$

$$= E_x[\sum_{n=0}^{\infty} c(X_n)\cdot W_n] .$$

Using an identical argument on the other two terms in equation (13), the result of the lemma is obtained.

(3.2) LEMMA: For a fixed control limit α, the long-term cost per unit time is given by

$$\psi_\alpha = \frac{f(x) + \int_{y\in[0,\alpha]} R(x,dy)[P\tilde{f}(y)-\tilde{f}(y)]}{E[\sigma] + \int_{y\in[0,\alpha]} R(x,dy)m(y)} ,$$

where

$$\tilde{f}(x) = f(x) - \int_{y\in E} R(x,dy)c(y)m(y)$$

and

$$P\tilde{f}(x) = \int_{y\in E} P(x,dy)\tilde{f}(y) + f(\Delta)[1-P(x,E)] .$$

Proof: The denominator of equation (11) comes from Feldman (1976, Proposition 2.10). The numerator given by equation (12) uses the relation

$$E_x[f(X_\Lambda)] = f(x) + \int_{y<\alpha} R(x,dy)[Pf(y)-f(y)], \tag{17}$$

Lemma (3.1), and the substitution given by (15). The proof of equation (17) appears in Feldman (1977) but since that proof contains some errors, (17) will be derived below. Using a renewal theoretic type argument and the fact that $P(x,y)=0$ for $x>y$ (namely the system is not self repairing), it follows that

$$\begin{aligned} E_x[f(X_\Lambda)] &= f(\Delta)[1-P(x,E)] + \int_{y\geq\alpha} P(x,dy)f(y) \\ &\quad + \int_{y<\alpha} P(x,dy)E_y[f(X_\Lambda)] \\ &= \int_{y<\alpha} R(x,dy)\Big\{\int_{u\geq\alpha} P(y,du)f(u) \\ &\quad + f(\Delta)[1-P(y,E)]\Big\}. \end{aligned}$$

Add and subtract $\int_{u<\alpha} P(y,du)f(u)$, rearrange terms, and obtain

$$\begin{aligned} E_x[f(X_\Lambda)] &= \int_{y<\alpha} R(x,dy)Pf(y) \\ &\quad - \int_{u<\alpha} f(u)\int_{y<\alpha} R(x,dy)P(y,du). \end{aligned}$$

The proof is now completed by noting that

$$RP(x,u) = R(x,u) - 1_{[0,u]}(x).$$

(3.3) THEOREM: Let (X,T) be the Markov renewal process defined by equations (3) and (4) with $X_0 = x$ and let R(x,y) be an absolutely continuous function with respect to y for each $x \in E$. Then the optimal control limit policy is the α that satisfies the following equation:

$$\int_{y \in [0,\alpha]} R(x,dy)\{m(y)[P\tilde{f}(\alpha)-\tilde{f}(\alpha)]-m(\alpha)[P\tilde{f}(y)-\tilde{f}(y)]\}$$

$$= m(\alpha)f(x)-E[\sigma][P\tilde{f}(\alpha)-f(\alpha)] \; .$$

Proof: Follows from Lemma (3.2) by taking a derivative with respect to α.

(3.4) THEOREM: Let (X,T) be the Markov renewal process defined by equation (3) and (4) with the state space the set of non-negative integers, let $X_0=i$, and let $r(i,j) = R(i,j)-R(i,j-1)$. Then the optimal control limit policy is the smallest α such that

$$\sum_{k=0}^{\alpha-1} r(i,k) \{m(k)[P\tilde{f}(\alpha)-\tilde{f}(\alpha)]-m(\alpha)[P\tilde{f}(k)-\tilde{f}(k)]\}$$

$$\geq m(\alpha)f(i)-E[\sigma][P\tilde{f}(\alpha)-f(\alpha)] \; .$$

Proof: Follows from Lemma (3.2) by taking differences with respect to α.

In the use of Theorem (3.4), it should be pointed out that the terms r(i,j) are easy to compute since Q is upper triangular (see Feldman (1976, Section 4)). It should also be noted that equality is included in equation (9) so replacement occurs when the cumulative damage is greater than or equal to the control limit. Finally, in order to avoid a common computation error, it might be helpful to emphasize that the state space E does not contain the state Δ.

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The semi-Markov shock model represents a system where shocks occur at random points in time causing random magnitudes of damage in such a way that the cumulative damage to the system over time is a semi-Markov process. System failure can occur at any of the shock times and the probability of failure is a function of the cumulative damage. An optimal control limit type policy has previously been derived assuming that costs are incurred at the replacement time This note is to show that the inclusion of state dependent maintenance costs does not significantly affect the form of the optimal control limit.

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